On C^r−closing for flows on 2-manifolds.

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Abstract

For some full measure subset $\mathcal{B}$ of the set of *iet*'s (i.e. interval exchange transformations) the following is satisfied:

Let X be a C^r, $1 \leq r \leq \infty$, vector field, with finitely many singularities, on a compact orientable surface M. Given a nontrivial recurrent point $p \in M$ of X, the holonomy map around p is semi-conjugate to an *iet* $E : [0,1) \to [0,1)$. If $E \in \mathcal{B}$ then there exists a C^r vector field Y, arbitrarily close to X, in the C^r−topology, such that Y has a closed trajectory passing through p.

1 Introduction

The open problem "C^r-closing lemma" is stated as follows:

"Let M be a smooth compact manifold, $r \geq 2$ be an integer, $f \in \mathrm{Diff}^r(M)$ (resp. $X \in \mathfrak{X}^r(M)$) and p be a nonwandering point of f (resp. of X). There exists $g \in \mathrm{Diff}^r(M)$ (resp. $Y \in \mathfrak{X}^r(M)$) arbitrarily close to f (resp. to X) in the C^r−topology so that p is a periodic point of g (resp. of Y)".

C. Pugh proved the C^1−closing lemma [Pg1]. There are few previous results when $r \geq 2$: Gutierrez [Gu1] showed similar results to this paper when the manifold is the torus T^2. There are negative answers: Gutierrez [Gu3] proved that if the perturbation is localized in a small neighborhood of the nontrivial recurrent point, then C^2−closing is not always possible. C. Carroll's [Car] proved that, even for flows with finitely many singularities, C^2−closing

by a twist-perturbation (supported in a cylinder) is not always possible. Concerning hamiltonian flows, M. Herman [Her] has remarkable counter-examples to the C^r-closing lemma. Within the context of geodesic laminations, S. Aranson and E. Zhuzhoma anounced in 1988 [A-Z] the C^r-closing lemma for a class of flows on surfaces; however, their proofs have not been published yet. For basic definition the reader may consult [K-H].

2 Statement of the results

Throughout this article, M will be a smooth, orientable, compact, two manifold and χ will be its Euler characteristic. We shall denote by $\mathfrak{X}^r(M)$ the space of vector field of class C^r, $1 \leq r \leq \infty$, with the C^r-topology. The trajectory of $X \in \mathfrak{X}^r(M)$ passing through $p \in M$ will be denoted by γ_p The domain of definition of a map S will be denoted by DOM(S). Smooth segments on M will be denoted and referred as (open, half-open, closed) intervals.

A bijective map $E : [0,1) \to [0,1)$ is said to be an *iet*, i.e. an *Interval Exchange Transformation* (with m intervals) if there exists a finite sequence $0 = a_1 < a_2 < \cdots < a_m < a_{m+1} = 1$ such that, for all $i \in \{1, 2, \cdots, m\}$ and for all $x \in [a_i, a_{i+1})$, $E(x) = E(a_i) + x - a_i$, and moreover, E is discontinuous at exactly $a_2, a_3, \cdots, a_m$. This E will be identified with the pair $(\lambda, \pi) \in \Delta_m \times \mathfrak{S}_m$ made up of the positive probability vector $\lambda = \{|a_{i+1} - a_i|\}_{i=1}^m$ and the permutation π on the symbols $1, 2, \cdots, m$, defined by $\pi(i) = \#\{j : E(a_j) \leq E(a_i)\}$. The space of *iet*'s, with m intervals, defined in $[0,1)$, will be identified with the *measurable space* $\Delta_m \times \mathfrak{S}_m$ endowed with the product measure, where Δ_m is the simplex of positive probability vectors of $I\!R^m$, with Lebesgue measure, and $\mathfrak{S}_m$ is the finite set of permutations on m symbols with counting measure. Let $E : [a,b) \to [a,b)$ be an *iet*. We say that $[s,t] \subset [a,b)$ is a *virtual orthogonal edge for* E, if E restricted to $[s,t]$ is continuous and $s < E(s) < E^2(s) = t$. Given $k \in I\!N$, let $\mathcal{B}_k$ be the set of *iet*'s $E : [a,b) \to [a,b)$ such that for some sequence $b_n \to a$ of points of (a,b), and for every $n \in I\!N$, the *iet* $E_n : [a, b_n) \to [a, b_n)$, induced by E, has at least $\chi + k + 3$, pairwise disjoint, virtual orthogonal edges. Denote $\mathcal{B} = \cap_{k \geq 1} \mathcal{B}_k$. It will be seen that, as a direct consequence of the work of W. A. Veech [Vee] and H. Masur [Mas],

Theorem 2.1. *For all $m \geq 2$, $\Delta_m \times \mathfrak{S}_m \setminus \mathcal{B}$ is a measure zero set.*

By transporting information along flow boxes, Item (a2) below follows from the definition of $\mathcal{B}_K$.

Theorem 2.2. *([Gu2, Structure Theorem, Section 3]) Let $X \in \mathfrak{X}^1(M)$. There are finitely many nontrivial recurrent trajectories $\gamma_{p_1}, \gamma_{p_2}, \cdots, \gamma_{p_\ell}$ of X such that if γ_p is any nontrivial recurrent trajectory of X, then $\overline{\gamma_p} = \overline{\gamma_{p_i}}$, for some $i = 1, 2, \cdots, \ell$.*

Suppose that X has exactly $K \in I\!N$ singularities (K=0 is allowed). Let $p \in M$ be a nontrivial recurrent point of X. Take a half-open interval $[p, q) \subset M$ transversal to X, such that p is a cluster point of $\gamma_p \cap (p, q)$, Denote by $P_X : [p, q) \to [p, q)$ the forward Poincaré map induced by X. If $[p, q)$ is small enough, it can be associated to $(p, [p, q))$, an iet $E = E_{(p,[p,q))} : [0, 1) \to [0, 1)$ and a continuous monotone surjective map $h : [p, q) \to [0, 1)$ such that $h(p) = 0$, h restricted to any given orbit of P_X is injective and, for all $x \in DOM(P_X)$, $E \circ h(x) = h \circ P_X(x)$; moreover,

- *(a1) there exists a subset $S \subset [0, 1)$ of at most $\chi + K + 2$ elements such that if A is a connected component of $[0, 1) \setminus S$, then $h^{-1}(A)$ is contained in $DOM(T)$;*
- *(a2) Let $\overline{p} \in \overline{\gamma_p}$ be a nontrivial recurrent point of X and $(\overline{p}, [\overline{p}, \overline{q}))$ be a pair satisfying the same conditions as those of $[p, [p, q))$ above. Then the property that the iet $E_{(\overline{p},[\overline{p},\overline{q}))}$ belongs to $\mathcal{B}_K$ does not depend on $(\overline{p}, [\overline{p}, \overline{q}))$.*

Under conditions of theorem above and if $E \in \mathcal{B}_K$, any nontrivial recurrent point of $\overline{\gamma_p}$ is said to be of $\mathcal{B}_K-$*type. Our result is the combination of Theorems 2.1 - 2.3.*

Theorem 2.3. *Let $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$, have $K \geq 0$ singularities. Let $p \in M$ be a $\mathcal{B}_K-$type nontrivial recurrent point of X. Then there exists $Y \in \mathfrak{X}^r(M)$, arbitrarily close to X, having a closed trajectory passing through p.*

Related to this theorem (see [Gu2]), we have that: For any $E \in \mathcal{B}$, it can be constructed $Y \in \mathfrak{X}^\infty(S)$, for some surface S, having a nontrivial recurrent point p_0 such that item (a1) is satisfied for some $h : [p_0, q_0) \to [0, 1)$, and $P_Y : [p_0, q_0) \to [p_0, q_0)$. Here, P_Y can be obtained to be injective or not.

3 Proof of the results

Suppose that M is endowed with an orientation and with a smooth riemannian metric $< , >$. Given a $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$, we define $X^\perp \in \mathfrak{X}^r(M)$

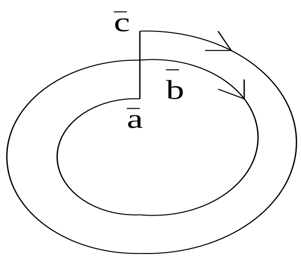

Fig. 1.a

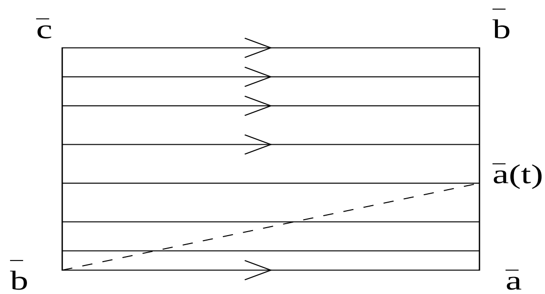

Fig. 1.b

by the following conditions: (a) $< X, X >=< X^{\perp}, X^{\perp} >$; and (b) when $p \in M$ is regular point of X, the ordered pair $(X(p), X^{\perp}(p))$ is an orthogonal positive basis of $T_p(M)$ (according to the given orientation of M). let Σ be an arc of trajectory of $X^{\perp}$. A $\Sigma-$*flow-box (for X)* is a compact subset $F \subset M$ whose interior is a flow box of X and whose boundary ∂F is a graph, homeomorphic to the figure "8", which is the union of arcs of trajectory $[\overline{c}, \overline{a}]_X$ and $[\overline{a}, \overline{c}]$ (connecting $\overline{a}$ and $\overline{c}$) of X and $X^{\perp}$, respectively. We shall refer to $[\overline{a}, \overline{c}]$ (resp. $[\overline{c}, \overline{a}]_X$) as the *orthogonal* (resp. *tangent) edge* of either F or ∂F. See Figs. 1.a and 1.b.

Let $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$, and let $p \in M$ be a nontrivial recurrent point of X. We say that X is *T-closable* at p (i.e. twist-closable at p) if there exists a half-open interval $\Sigma = [p, q)$ tangent to $X^{\perp}$, such that, for any neighborhood V of p, there exists a $\Sigma-$flow-box for X having its orthogonal edge contained in $\Sigma \cap V$.

Proposition 3.1. *Let $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$, and let $p \in M$ be a non-trivial recurrent point of X. Suppose that X is T-closable at p. Then there are sequences $t_n \to 0$, of real numbers, and $p_n \to p$, of points of M, such that $X + t_n X^{\perp}$ has a closed trajectory through p_n*

Proof: As X is T-closable at p, there exists a half-open interval $\Sigma = [p, q)$ tangent to $X^{\perp}$, such that, given neighborhoods $\mathcal{V}$ of X and V of p, we may choose a $\Sigma-$flow-box $F \subset M$ (for X) and $\sigma > 0$ such that if $[\overline{c}, \overline{a}]_X$ and $[\overline{a}, \overline{c}])$ are the tangent and orthogonal edges, respectively, of ∂F, and $\overline{b}$ is the vertex of ∂F, then:

(b1) $[\overline{a}, \overline{c}] \subset V$ and the flow of X enters into F through the closed subinterval $[\overline{b}, \overline{c}]$ of Σ; moreover, for all $t \in [-\sigma, \sigma]$, $X(t) := X + t\, X^{\perp} \in \mathcal{V}$;

(b2) both $X(\sigma)$ and $X(-\sigma)$ have an arc of trajectory contained in F, which is a global cross section for $X|_F$.

We shall continue considering only the case in which the flow of $X^\perp$ goes from $\overline{a}$ to $\overline{c}$. Let Γ be the set of real numbers $s \in [0,\sigma]$ such that when $t \in [0,s]$ there is an arc of trajectory $[\overline{b}, \overline{a}(t)]_{X(t)}$ of $X(t)$, joining $\overline{b}$ with $\overline{a}(t) \in [\overline{a}, \overline{b}]$, contained in F, with $\overline{a}(0) = \overline{a}$, and such that $\overline{a}(t)$ depends continuously on t. When $t \in \Gamma$, these conditions determine $\overline{a}(t)$ and also that $[\overline{b}, \overline{a}(t)]_{X(t)}$ is transversal to X. Therefore, by (b2), $\Gamma = [0, \sigma_1]$ is a closed interval, $\overline{a}(\sigma_1) = b$ and $[\overline{b}, \overline{a}(\sigma_1)]_{X(\sigma_1)}$ is a closed trajectory of $X(\sigma_1)$. See Fig. 1.b $\square$

Under the assumptions and conclusions of this proposition, there exists a sequence $F_n : M \to M$ of C^r-diffeomorphisms, taking p_n to p. We may assume that F_n converges to the identity diffeomorphism in the $C^{r+1}-$topology. Therefore, the sequence of vector fields $(F_n)_*(X + t_n X^\perp) \to X$ in the C^r-topology and each $(F_n)_*(X + t_n X^\perp)$ has a closed trajectory passing through p. This proves the following

Theorem 3.2. *Let $X \in \mathfrak{X}^r(M)$, $1 \le r \le \infty$. Let $p \in M$ be a nontrivial recurrent point of X. Suppose that that X is T-closable at p. Then there exists $Y \in \mathfrak{X}^r(M)$ arbitrarily close to X having a closed trajectory through p.*

Proof of Theorem 2.1: We shall prove that: For all $m \ge 2$, $\Delta_m \times \mathfrak{S}_m \setminus \mathcal{B}$ is a measure zero set. It was proved by W. A. Veech [Vee] and H. Masur [Mas] that the Rauzy operator $\mathcal{R} : \mathcal{M} \to \mathcal{M}$, defined in a full measure subset $\mathcal{M}$ of $\Delta_m \times \mathfrak{S}_m$, is ergodic and has the following property:

(c) Given $E \in \mathcal{M}$, there exists a sequence $\{[0, a_n)\}$ of subintervals of $[0,1)$ such that $a_n \to 0$ and, if $\tilde{E}_n : [0, a_n) \to [0, a_n)$ denotes the *iet* induced by E, then, up to re-scaling, $\mathcal{R}^n(E)$ coincides with $\tilde{E}_n$; more precisely, $\mathcal{R}^n(E)(z) = (1/a_n)\tilde{E}_n(a_n z)$, for all $z \in [0,1)$.

Given $k \ge 1$, let A_k be the set of $E \in \Delta_m \times \mathfrak{S}_m$ such that for some $a \in (16^{-k} - 32^{-k}, 16^{-k} + 32^{-k})$, $E(x) = a + x$, for all $x \in [0, 1/2]$. We observe that A_k is open and so it has positive measure. Let $\tilde{\mathcal{B}}_k$ be the set of $E \in \mathcal{M}$ such that the positive $\mathcal{R}-$orbit of E visits A_k infinitely many often. As A_k has positive measure and $\mathcal{R}$ is ergodic, the complement of $\tilde{\mathcal{B}}_k$ has measure zero. Therefore, the complement of $\tilde{\mathcal{B}} = \cap_{k \ge 2} \tilde{\mathcal{B}}_k$ has measure zero. Observe that if and *iet* $E \in A_k$, then E has more than k, pairwise disjoint, virtual orthogonal edges. Therefore, as $\mathcal{R}$ satisfy (c) right above and since the positive $\mathcal{R}-$orbit

of any given $E \in \tilde{\mathcal{B}}$ visits every A_k infinitely many often, we obtain that $\tilde{\mathcal{B}} \subset \mathcal{B}$. this proves the theorem. $\square$

Proof of Theorem 2.3: This theorem is stated as follows: Let $p \in M$ be a $\mathcal{B}_K-$type nontrivial recurrent point of $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$. Suppose that X has $K \geq 0$ singularities. Then there exists a $Y \in \mathfrak{X}^r(M)$ arbitrarily close to X, having a closed trajectory passing through p.

By theorem 3.2, it is enough to prove that X is T-closable at p. Let $\Sigma = [p,q)$, $T : [p,q) \to [p,q)$, $E : [0,1) \to [0,1)$, $h : [p,q) \to [0,1)$ be as in Theorem 2.2. As $E \in \mathcal{B}_K$, given a neighborhood V of p, there exist $b \in (0,1)$ and an *iet* $E_V : [0,b) \to [0,b)$, such that:

(e) E_V has at least $\chi + K + 3$ pairwise disjoint virtual orthogonal edges contained in $[0,b)$; moreover, the interval $\Sigma_V = h^{-1}([0,b))$ is contained in V.

Let $T_V : \Sigma_V \to \Sigma_V$ be the map induced by T. As X has K singularities, (e) and Theorem 2.2 imply that E_V has a virtual orthogonal edge $[a, E_V(a)] \subset [0,b)$ such that, for some $\overline{a} \in \mathrm{DOM}((T_{\Sigma_V}))$, $[\overline{a}, T_V(\overline{a})] = h^{-1}([a, E(a)]) \subset \mathrm{DOM}(T|_{\Sigma_V})$. Therefore, there exists a $\Sigma-$flow-box bounded by $[\overline{a}, T_V^2(\overline{a})] \cup [\overline{a}, T_V^2(\overline{a})]_X$. As V is arbitrary, this proves that X is T-closable at p. $\square$

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